SIMULATION OF A REACTION WHEEL INVERTED PENDULUM

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Abstract: *The paper describes numerical simulation of the motion of an one-dimensional pendulum with a reaction wheel. Three laws for controlling both the pendulum and the wheel have been studied successively. The goal is to balance the pendulum in an upright position. The control loop includes proportional and derivative term. Governing differential equations are solved by means of block diagrams in SciLab / xCos environment. The block diagrams are published, so are the numerical results.*

СИМУЛАЦИЯ НА ДВИЖЕНИЕТО НА МАХАЛО С ИНЕРЦИОНЕН МАХОВИК

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Ключови думи: *реактивен / инерционен маховик, махало, PID контролер, SciLab / xCos*

Резюме: *В доклада е описана числена симулация на движението на махало с инерционен маховик. Изучени са последователно три закона за управление на махалото и маховика. Целта е да се балансира махалото в изправено положение. Контурът за управление включва пропорционално и диференциращо звено. Диференциалните уравнения, описващи движението, са решени с блокови схеми в среда SciLab / xCos. Публикувани са блок-схемите, както и получените числени резултати.*

Introduction

Task of satellite attitude control is commonly accomplished by means of the so-called reaction wheel(s), i.e. a flywheel giving the satellite a stored amount of angular momentum. The conservation law implies that total angular momentum of a mechanical system remains unchanged on condition that an external torque is not applied. Hence, the satellite counter-rotates according to changes in wheel angular velocity vector. This concept of attitude control is solely applicable to small satellites.

This paper makes reference to the work of Block et al. who did an innovative research on reaction wheel pendulum and published the results in monography [1]. Authors set out a powerful mathematical apparatus to describe both pendulum and wheel dynamics and go further into great detail to support their assertions. Both linearized and non-linear pendulum models are examined fully. A connection between PID controller coefficients and system stability is established. Exemplary problems are solved and discussed for different cases of pendulum / wheel control.

The current research draws an analogy between experimental and numerical results by proposing a feasible alternative to real pendulum and wheel dynamics. To work out a numerical solution to linearized equations governing both pendulum and wheel is a primary goal of the presented study. The equations are proposed by Block et al. in monography [1]. Three study cases are thoroughly investigated:

- a simple proportional-derivative (PD) control of pendulum angle;
- a PD control of pendulum angle and wheel angular velocity;
- a PD control of pendulum angle and wheel angle.

The experiment was implemented in SciLab / xCos, [2] environment.

Materials and Methods

In Fig. 1, a two-dimensional projection view is shown alongside a three-dimensional perspective view of both wheel and pendulum under consideration.

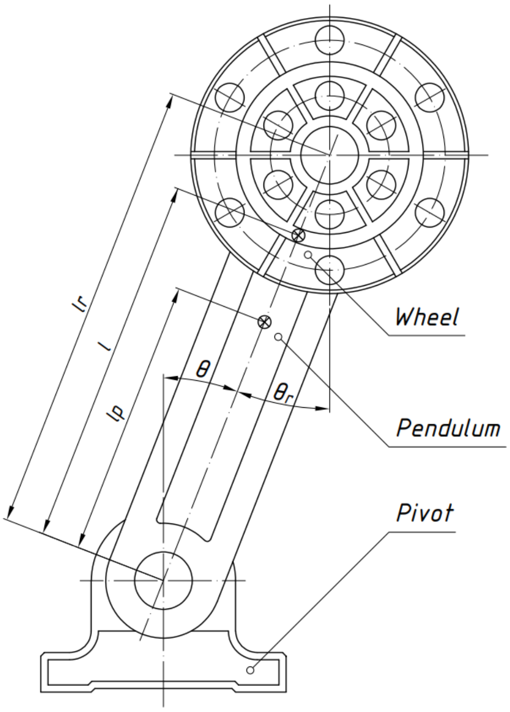

Fig. 1a. Experiment setup

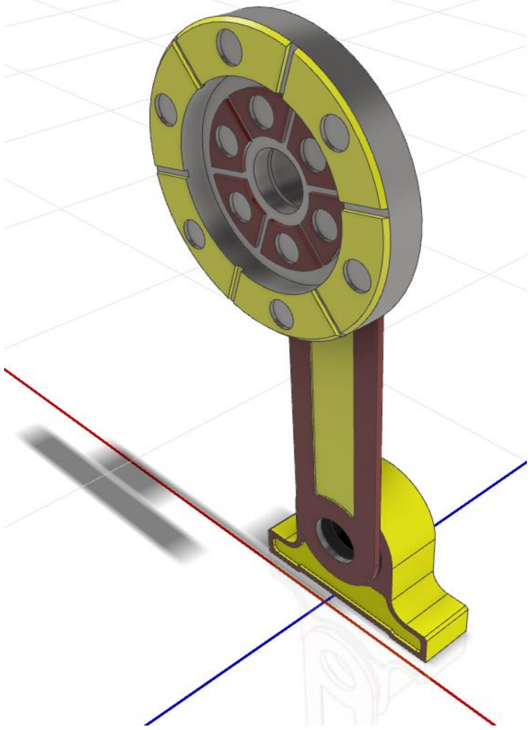

Fig. 1b. Inverted pendulum 3D

The experiment setup consists of a pivot, an inverted pendulum, and a reaction wheel, Fig. 1a. In order that pendulum maintains upright stance, the wheel should rotate in accordance with a control law. Throughout the numerical experiment, values of pendulum angle and wheel revolutions / angle are fed back to the system input. Actuating and sensing devices are neglected which is why the results are less likely to depend on angle (or angular velocity) sensor resolution, for instance a rotary encoder. Driving motors, cables are also omitted to simplify the experiment further. The system mass properties resemble those quoted in monography [1] as closely as possible.

Governing Equations

The governing ordinary differential equations (ODE) are borrowed from monography [1] in the form they were derived. The model describing system dynamics is following

(1)
$$\ddot{\theta} + a.\sin(\theta) = -b_p(u - F)$$
$$\ddot{\theta}_r = b_r(u - F)$$

where **θ** is pendulum deflection angle, **θ_r** is wheel deflection angle, **u** is control input, **F** is friction torque on the motor axis, **a**, **b_p**, **b_r** are constants depending on system mass properties and DC motor current, [1]. In current research, friction is neglected, equations are linearized about the pendulum upright position **$\theta = \pi$**. Therefore, both pendulum and wheel obey the following simplified law:

(2)
$$\ddot{\theta} - a.\theta = -b_p u$$
$$\ddot{\theta}_r = b_r u$$

By introducing a control law of pendulum angle **θ** (simple PD controller)

(3)
$$u = -k_{pp}\theta - k_{dp}\dot{\theta}$$

and subsequently inserting in Eq. (2), following equations and SciLab / xCos subsystem, Fig. 2, are obtained:

(4)
$$\ddot{\theta} - b_p k_{dp}\dot{\theta} - (b_p k_{pp} + a)\theta = d \quad \theta(0) = 0 \qquad \dot{\theta}(0) = 0$$
$$\ddot{\theta}_r + b_r k_{dp}\dot{\theta} + b_r k_{pp}\theta = 0 \quad \theta_r(0) = const \qquad \dot{\theta}_r(0) = 0$$

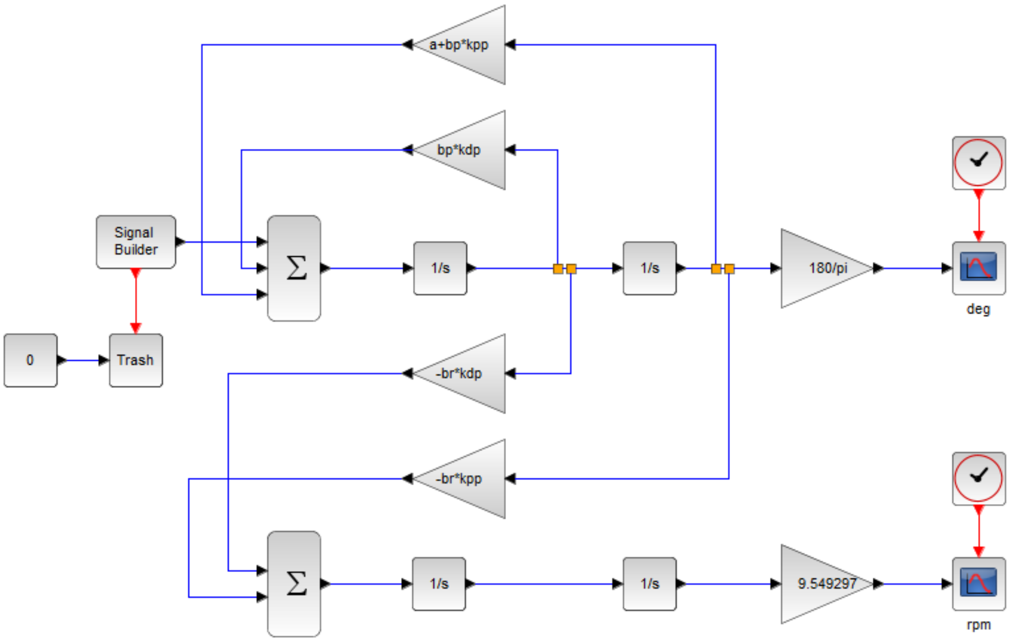

Fig. 2. Subsystem for working out a solution to system ODEs (4) (pendulum angle)

Alternatively, after applying a control law of pendulum angle **θ** and wheel angular velocity **ω = dθ$_r$/dt**

(5) $$u = -k_{pp}\theta - k_{dp}\dot{\theta} + k_{dr}\left(\omega_0 - \omega\right)$$

governing equations and SciLab / xCos subsystem are obtained as follows:

(6) $$\begin{aligned} &\ddot{\theta} - b_p k_{dp}\dot{\theta} - \left(a + b_p k_{pp}\right)\theta - b_p k_{dr}\omega = -b_p k_{dr}\omega_0 + d \quad \theta(0)=0 \quad \dot{\theta}(0)=0 \\ &b_r\left(k_{dp}\dot{\theta} + k_{pp}\theta\right) + \dot{\omega} + b_r k_{dr}\omega = b_r k_{dr}\omega_0 \qquad \omega(0)=const \end{aligned}$$

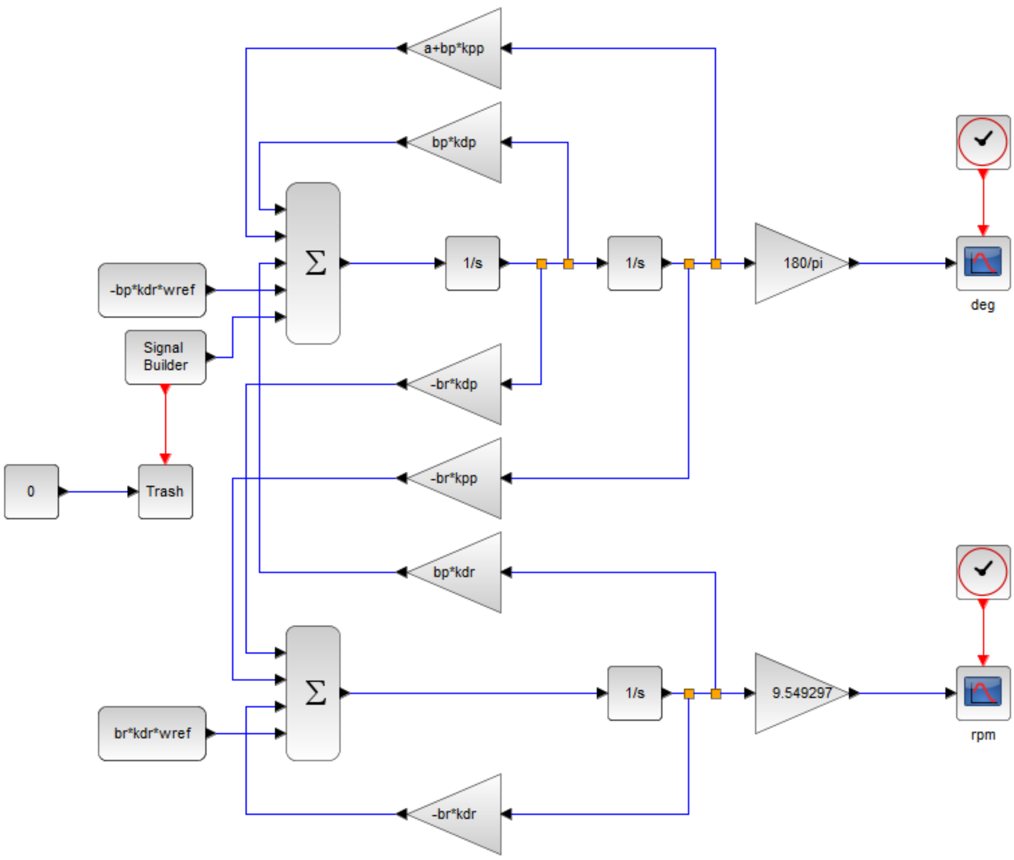

Fig. 3. Subsystem for working out a solution to system ODEs (6) (wheel angular velocity)

Finally, following control law of pendulum angle **θ** and wheel angle **θ$_r$**

(7) $$u = k_{pp}\left(\theta_0 - \theta\right) - k_{dp}\dot{\theta} + k_{pr}\theta_r - k_{dr}\dot{\theta}_r$$

results in governing equations

(8) $$\begin{aligned} &\ddot{\theta} - b_p k_{dp}\dot{\theta} - \theta\left(a + b_p k_{pp}\right) - b_p\left(k_{dr}\dot{\theta}_r + k_{pr}\theta_r\right) = -b_p k_{pp}\theta_0 + d \quad \theta(0)=0 \quad \dot{\theta}(0)=0 \\ &b_r\left(k_{dp}\dot{\theta} + k_{pp}\theta\right) + \ddot{\theta}_r + b_r k_{dr}\dot{\theta}_r + b_r k_{pr}\theta_r = b_r k_{pp}\theta_0 \quad \theta_r(0)=0 \quad \dot{\theta}_r(0)=0 \end{aligned}$$

In equations above, **d** (1 if t ∊ [3; 4); 0 otherwise) stands for a disturbance torque.

The SciLab / xCos subsystem, corresponding to eq. (8), is shown in Fig. 4:

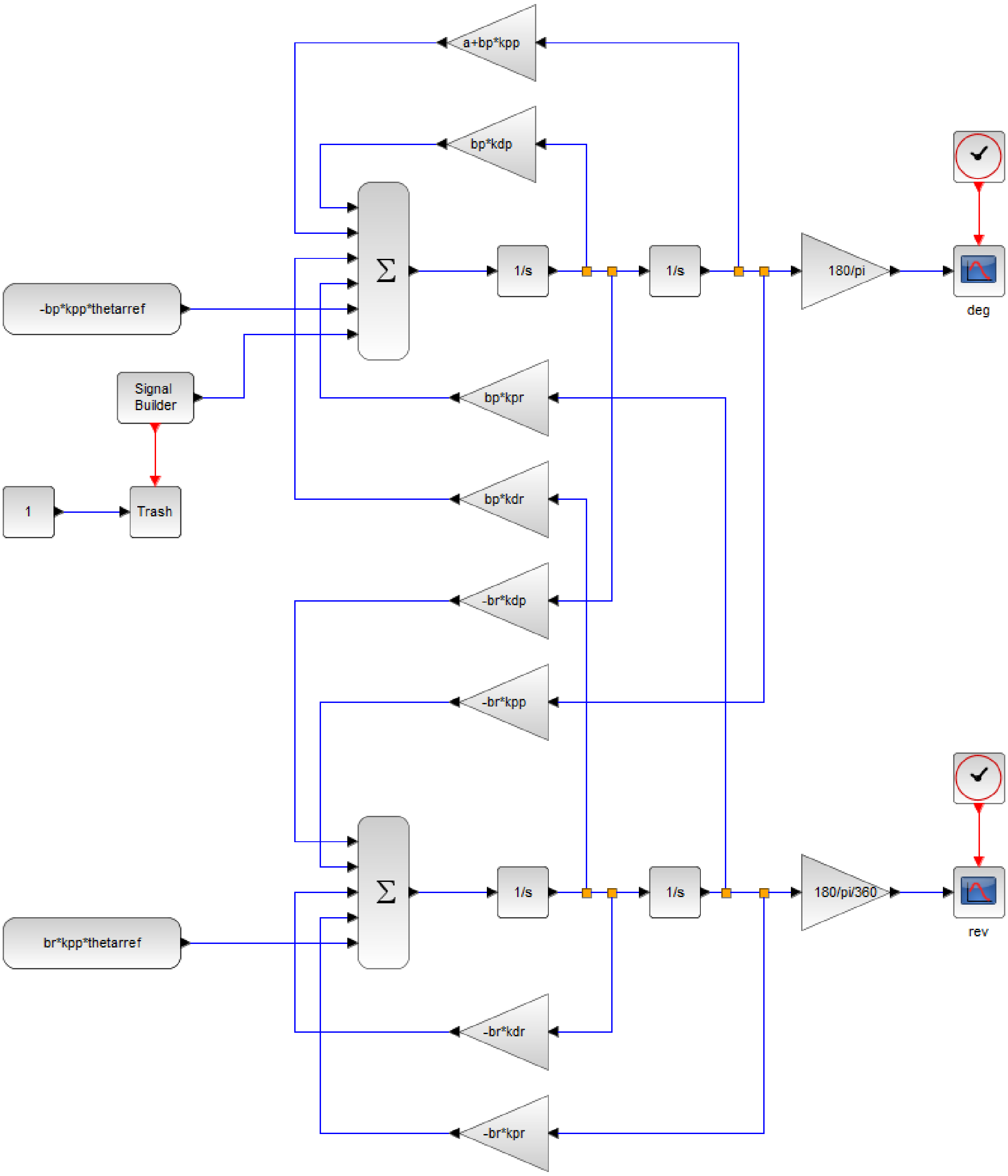

Fig. 4. Subsystem for working out a solution to system ODEs (8) (wheel angle)

Controller coefficients are derived taking into account characteristic polynomials of systems ODEs (4), (6), and (8). Coefficients values depend on pendulum oscillation frequency and system damping ratio. Values obtained in current study are following:

- Generic coefficients and mass properties

a = 78.908; bp = 0.9537; br = 207.11; mr = 0.265 kg; mp = 0.092 kg; Jp = 211.662e-06 kg.m^2; Jr = 238.134e-06 kg.m^2; lp = 86.32e-03 m; lr = 127e-03 m; I = 116.5, (Fig. 1a)

- Pendulum angle: kpp = –268.90; kdp = –19.758; ς = 0.707
- Wheel angular velocity: kdr = –0.028951; kpp = –321.55; kdp = –28.839; wref =10.471976 rad/s = 100 rpm; ω0 = 13.325; ς = 0.707; α = 0.2
- Wheel angle: kpp = –564.93; kdp = –63.597; kpr = –0.3810; kdr = –0.1464; θref = 0; ω1 = 8.883; ς1 = 0.707; ω2 = 8.883; ς2 = 1

Details on how to compute coefficients might be found in monography [1].

Mass and inertial properties of both pendulum and rotor were computed taking into account density of stainless steel, 8 g/cm^3. Both linear and angular dimensions are shown in Fig. 1a along with mass centres locations and part names.

Results

Following results are obtained for three study cases as follows.

- Simple PD controller (Fig. 2) of pendulum angle only

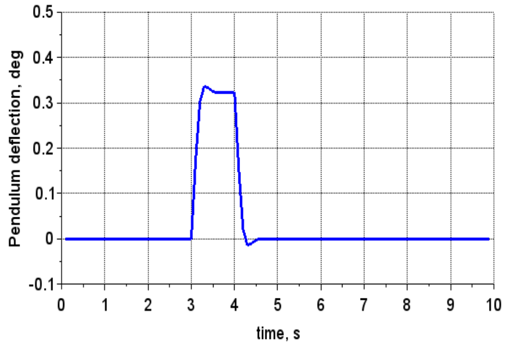

Fig 5a. Pendulum deflection, deg, vs. time, s

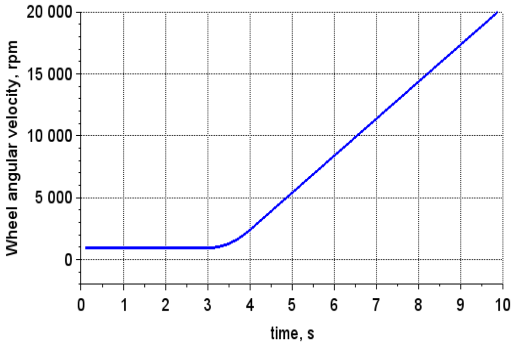

Fig. 5b. Wheel angular velocity, rpm, vs. time, s

- PD controller (Fig. 3) of pendulum angle and wheel angular velocity

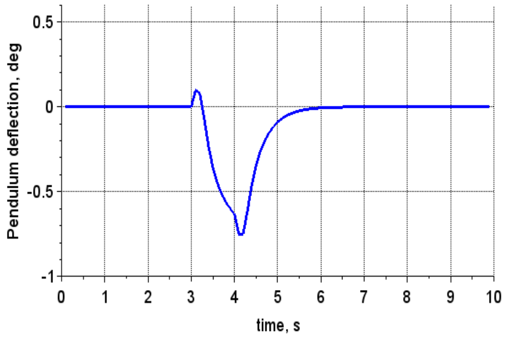

Fig. 6a. Pendulum deflection, deg, vs. time, s

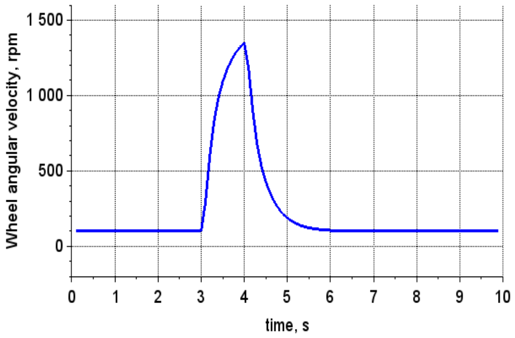

Fig. 6b. Wheel angular velocity, rpm, vs. time, s

- PD controller (Fig. 4) of pendulum angle and wheel angle

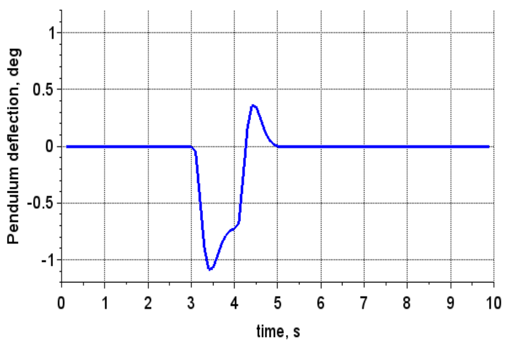

Fig. 7a. Pendulum deflection, deg, vs. time, s

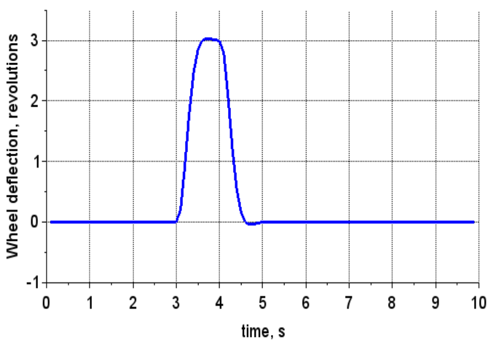

Fig. 7b. Wheel deflection, revolutions, vs. time, s

Discourse

Consider Fig. 5. Although the pendulum is able to restore initial upright attitude, the wheel angular speed reaches saturation limit fast. At the onset of a disturbance torque, the wheel starts accelerating at constant rate. This result was discussed in monography [1] and confirmed by the numerical experiment. Feedback from wheel angle / angular speed is not introduced by control law (3) whatsoever (open loop), hence the (intuitively predictable) results.

Consider Fig. 6. Initial values of angular velocity $\boldsymbol{\omega = d\theta_r/dt}$ imply that the wheel maintains 100 rpm. The wheel rotates about 10 times faster at the end of a disturbance pulse applied to the pendulum. After the disturbance torque ceases, the wheel restores value of initial angular rate.

Consider Fig. 7. During the disturbance torque, the wheel rotates three revolutions in both directions. After the disturbance torque comes to an end, the wheel restores initial angle of deflection.

It should be noted that Fig. 5a, 6a, and 7a depict pendulum deviations $\boldsymbol{\delta\theta = \theta - \pi}$. The problem is well posed and the solution is correct so long as disturbance torque **d** is small.

Alternatively, source code in Appendix might be used if block diagrams are less preferable.

References:

1. Block, D. J., K. J. Astrom, M. W. Spong, The Reaction Wheel Pendulum, Synthesis Lectures on Control and Mechatronics, Lecture #1, 2007 by Morgan & Claypool.
2. https://www.scilab.org/software/xcos
3. https://octave.org/

Appendix. Source code for solving Eq. (4)

GNU Octave, [3]

```
clear;

function [pulse] = d(t)

  if (t < 3 || t >= 4)
    pulse = 0;
  else
    pulse = 1;
  end

end

function [dx] = sys(t, x)

  % Parameters
  a = 78.908; bp = 0.9537; br = 207.11;

  % PID tuning
  kpp = -268.90; kdp = -19.758;

  dx = zeros(4, 1);

  A =
[0,1,0,0;a,0,0,0;0,0,0,1;0,0,0,0];
  B = [0; -bp; 0; br;];
  K = [-kpp, -kdp, 0, 0];
  u = K*x;
  dx = A*x + B*u + [0; d(t); 0; 0];

end

tspan = 0:0.1:10;
x0 = [0; 0; 100; 0];
[t, x] = ode45(@sys, tspan, x0);

figure(1); % theta
plot(t, x(:, 1) * 180/pi, 'g');
grid on

figure(2); % theta_r
plot(t, x(:, 3) * 9.549297, 'm');
grid on
```

SciLab, [2]

```
clear;

function [pulse] = d(t)

  if (t < 3 || t >= 4)
    pulse = 0;
  else
    pulse = 1;
  end

end

function [dx] = sys(t, x)

  // Parameters
  a = 78.908; bp = 0.9537; br = 207.11;

  // PID tuning
  kpp = -268.90; kdp = -19.758;

  dx = zeros(4, 1);

  A = [0,1,0,0;a,0,0,0;0,0,0,1;0,0,0,0];
  B = [0; -bp; 0; br];
  K = [-kpp, -kdp, 0, 0];
  u = K*x;
  dx = A*x + B*u + [0; d(t); 0; 0];

end

tspan = 0:0.1:10;
x0 = [0; 0; 100; 0];
x = ode("rkf", x0, 0, tspan, sys);

figure(1); // theta
plot(tspan, x(1, :) * 180/%pi);
xgrid(1, 1, 7);

figure(2); // theta_r
plot(tspan, x(3, :) * 9.549297);
xgrid(1, 1, 7);
```